

NEWTONIAN LIMIT OF MAXWELL FLUID FLOWS

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ABSTRACT. In this paper, we revise Maxwell's constitutive relation and formulate a system of first-order partial differential equations with two parameters for compressible viscoelastic fluid flows. The system is shown to possess a nice conservation-dissipation (relaxation) structure and therefore is symmetrizable hyperbolic. Moreover, for smooth flows we rigorously verify that the revised Maxwell's constitutive relations are compatible with Newton's law of viscosity.

1. INTRODUCTION

Maxwell fluids are among macromolecular or polymeric fluids. A large number of experiments indicate that polymeric fluids exhibit elastic as well as viscous properties [1]. Thus, they are quite different from small molecular fluids. The latter have viscosity as the main feature, are satisfactorily characterized by Newton's law of viscosity

$$\tau = -\nu \left[\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] - \kappa \nabla \cdot v I, \quad (1.1)$$

and are also called Newtonian fluids. Here $\tau = \tau(x, t)$ is the stress tensor of the fluid at space-time (x, t) , ν is the shear viscosity, κ is the bulk viscosity, $v = v(x, t)$ is the velocity, ∇ is the gradient operator with respect to the space variable $x = (x_1, x_2, x_3)$, the superscript T stands for the transpose operator, and I denotes the unit matrix of order 3. Combining Newton's law of viscosity with the conservation laws of mass, momentum and energy, one gets the classical Navier-Stokes equations.

To account for the elastic properties of polymeric fluids, Maxwell combined Newton's law of viscosity with Hooke's law of elasticity and proposed the following constitutive relation [8]

$$\epsilon \tau_t + \tau = -\nu \left[\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] - \kappa \nabla \cdot v I. \quad (1.2)$$

Here ϵ is the ratio of the viscosity and the elastic modulus. A Maxwell fluid is that obeying the constitutive relation (1.2). This relation reflects that the stress tensor responds to the fluid motion in a delayed, instead of instant, fashion. It has motivated many more realistic and nonlinear constitutive relations, including the well-known upper-convected Maxwell (UCM) and Oldroyd-B models [3].

In this paper, we revise Maxwell's constitutive relation (1.2), combine the conservation laws and formulate the following partial differential equations for compressible viscoelastic

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fluid flows:

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho v) &= 0, \\
\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v + pI) + \frac{1}{\epsilon_1} \nabla \cdot \tau_1 + \frac{1}{\epsilon_2} \nabla \tau_2 &= 0, \\
\partial_t \tau_1 + \frac{1}{\epsilon_1} [\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I] &= -\frac{\tau_1}{\nu \epsilon_1^2}, \\
\partial_t \tau_2 + \frac{1}{\epsilon_2} \nabla \cdot v &= -\frac{\tau_2}{\kappa \epsilon_2^2}.
\end{aligned} \tag{1.3}$$

Here ρ is the density of the fluid, \otimes denotes the tensorial product, $p = p(\rho)$ is the hydrostatic pressure, τ_1 is a tensor of order two, ϵ_1 and ϵ_2 are two positive parameters, and τ_2 is a scalar. This is a system of first-order partial differential equations, with domain

$$G := \{(\rho, \rho v, \tau_1, \tau_2) : \rho > 0\}.$$

In (1.3) there are 14 equations (for three-dimensional problems). Note that the $[\dots]$ -term in the τ_1 -equation is symmetric and traceless. It is easy to see that τ_1 is symmetric and traceless if it is so initially. When τ_1 is symmetric and traceless, the number of independent equations in (1.3) reduces to $n = 10$. Throughout this paper, we assume that τ_1 is symmetric and traceless.

We will show that the first-order system (1.3) satisfies the entropy dissipation condition proposed in [13]. This particularly implies that the system is symmetrizable hyperbolic. Moreover, we will show that the revised Maxwell's constitutive relations (the τ_1 -, τ_2 -equations) in (1.3) are compatible with Newton's law of viscosity (1.1) for small ϵ_1 and ϵ_2 . To see this, we rewrite the two τ -equations in (1.3) as follows and iterate them once to obtain

$$\begin{aligned}
\tau_1 &= -\epsilon_1 \nu [\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I] - \epsilon_1^2 \nu \partial_t \tau_1, \\
&= -\epsilon_1 \nu [\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I] + O(\epsilon_1^3), \\
\tau_2 &= -\epsilon_2 \kappa \nabla \cdot v - \epsilon_2^2 \kappa \partial_t \tau_2, \\
&= -\epsilon_2 \kappa \nabla \cdot v + O(\epsilon_2^3).
\end{aligned}$$

Substituting the truncations into the momentum equation in (1.3), we obtain the classical isentropic Navier-Stokes equations. In this sense, Newton's law (1.1) is recovered.

A major part of this paper is devoted to a rigorous justification of the compatibility above. To do this, we employ the convergence-stability principle [12, 2] for initial-value problems of symmetrizable hyperbolic systems and prove that, as ϵ_1 and ϵ_2 go to zero, smooth solutions to the first-order system exist in the time interval where the isentropic Navier-Stokes equations have smooth solutions and converge to the latter. Namely, we show that the first-order system (1.3) is a diffusive relaxation approximation to the isentropic Navier-Stokes equations.

Let us remark that, despite being quite similar, the present problem is very different from those studied in [5, 14]. In fact, when writing (1.3) in its quasilinear form, the coefficients of $\frac{1}{\epsilon}$ in the left-hand side depend on ρ and ρv . Therefore, our problem does not possess the parabolic structure required in [5]. It also differs from that in [11], for ρ and ρv are not dissipative quantities. Because of these, our analysis contains some innovative treatments relying on the specific structure of (1.3).

We end this introduction by mentioning some other related works known to the author. Studying the motion of complex fluids involves many challenging and interesting partial differential equations [10] and has attracted much attention in recent years (see [7] and references cited therein). Most mathematical literature are concerned with well-posedness of incompressible flows governed by partial differential equations with upper-convected derivatives included. It seems that there are very few results on the Newtonian limit of non-Newtonian fluid flows. The only one known to this author is [9], which was concerned with incompressible viscoelastic fluid flows of Oldroyd type for solutions in the Besov spaces.

The paper is organized as follows. In Section 2 we show that the first-order system (1.3) satisfies the entropy dissipation condition proposed in [13]. Section 3 is devoted to a precise statement of our compatibility result. A key error estimate is derived in Section 4.

2. ENTROPY DISSIPATION STRUCTURE

In this section, we show that the first-order system (1.3) satisfies the entropy dissipation condition proposed in [13]. To this purpose, we define

$$\Phi(\rho) = \rho \int_1^\rho \frac{p(z)}{z^2} dz$$

and compute, for smooth solutions to (1.3),

$$\begin{aligned} & \partial_t (4\Phi(\rho) + 2\rho|v|^2 + 2\tau_2^2 + |\tau_1|^2) \\ & + \nabla \cdot (4\Phi(\rho)v + 2\rho|v|^2v + 4pv + 4\frac{\tau_2 v}{\epsilon_2} + 4\frac{\tau_1 v}{\epsilon_1}) = -\frac{4\tau_2^2}{\kappa\epsilon_2^2} - \frac{2|\tau_1|^2}{\nu\epsilon_1^2}. \end{aligned} \quad (2.1)$$

Here $|\tau_1|^2$ is the trace of the matrix $\tau_1^T \tau_1$ and we have used the fact that τ_1 is a symmetric and traceless tensor. It is easy to verify that

$$\eta = \eta(U) := 4\Phi(\rho) + 2\rho|v|^2 + 2\tau_2^2 + |\tau_1|^2$$

is a strictly convex function of $U := (\rho, \rho v, \tau_1, \tau_2)^T$, provided that the pressure $p = p(\rho)$ is strictly increasing with respect to $\rho > 0$. Thus, the first-order system (1.3) fulfills the entropy dissipation condition in [13]. This particularly implies that the system is symmetrizable hyperbolic.

Set

$$w = (\rho, \rho v)^T \quad \text{and} \quad z = (\tau_1, \tau_2)^T.$$

We may rewrite the first-order system (1.3) (with $\epsilon_1 = \epsilon_2 \equiv \epsilon$ for simplicity) as

$$\begin{aligned} \partial_t w + \sum_j f_j(w)_{x_j} + \frac{1}{\epsilon} \sum_j C_j z_{x_j} &= 0, \\ \partial_t z + \frac{1}{\epsilon} \sum_j g_j(w)_{x_j} &= -\frac{1}{\epsilon^2} S w, \end{aligned} \quad (2.2)$$

where C_j is a constant matrix and $S = \text{diag}(\nu^{-1}I_9, \kappa^{-1})$. Moreover, from the form of $\eta = \eta(U)$ and the strict convexity it follows that $\eta_{ww}(w, z)$ is a symmetric positive-definite matrix, $\eta_{zz}(w, z)$ is a constant diagonal (and positive-definite) matrix, $\eta_{ww}(w, z)f_{jw}(w)$ is symmetric, and

$$\eta_{ww}(w, z)C_j = g_{jw}(w)^T \eta_{zz}(w, z)^T. \quad (2.3)$$

The last two statements are based on (2.1).

3. COMPATIBILITY THEOREM

This section is devoted to a precise statement of our compatibility result. For the sake of simplicity, we assume that $\epsilon_1 = \epsilon_2 \equiv \epsilon$ in what follows.

Let $\rho = \rho(x, t)$ and $v = v(x, t)$ be the density and velocity of the Newtonian fluid. Then they obey the conservation laws of mass and momentum

$$\partial_t \rho + \nabla \cdot (\rho v) = 0,$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v + p(\rho)I) + \nabla \cdot \tau = 0$$

together with Newton's law of viscosity (1.1)

$$\tau = -\nu \left[\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right] - \kappa \nabla \cdot v I.$$

Namely, they solve the isentropic Navier-Stokes equations.

Define

$$\rho_\epsilon = \rho, \quad v_\epsilon = v, \quad \tau_{1\epsilon} = -\epsilon \nu \left[\nabla v + (\nabla v)^T - \frac{2}{3} \nabla \cdot v I \right], \quad \tau_{2\epsilon} = -\epsilon \kappa \nabla \cdot v.$$

We have

$$\begin{aligned} \partial_t \rho_\epsilon + \nabla \cdot (\rho_\epsilon v_\epsilon) &= 0, \\ \partial_t(\rho_\epsilon v_\epsilon) + \nabla \cdot (\rho_\epsilon v_\epsilon \otimes v_\epsilon + p(\rho_\epsilon)I) + \frac{1}{\epsilon} \nabla \cdot \tau_{1\epsilon} + \frac{1}{\epsilon} \nabla \cdot \tau_{2\epsilon} &= 0, \\ \partial_t \tau_{1\epsilon} + \frac{1}{\epsilon} \left[\nabla v_\epsilon + (\nabla v_\epsilon)^T - \frac{2}{3} \nabla \cdot v_\epsilon I \right] &= -\frac{\tau_1}{\nu \epsilon^2} + \partial_t \tau_{1\epsilon}, \\ \partial_t \tau_{2\epsilon} + \frac{1}{\epsilon} \nabla \cdot v_\epsilon I &= -\frac{\tau_2}{\kappa \epsilon^2} + \partial_t \tau_{2\epsilon}. \end{aligned} \tag{3.1}$$

Our compatibility result can be stated as

Theorem 3.1. *Suppose the pressure function $p = p(\rho)$ is strictly increasing with respect to $\rho > 0$, the density ρ and velocity v of the Newtonian fluid are continuous and bounded in $(x, t) \in \Omega \times [0, T_*]$ with $T_* < \infty$, and satisfy $\inf_{x,t} \rho(x, t) > 0$ and*

$$\nabla \rho \in C([0, T_*], H^s(\Omega)), \quad v \in C^1([0, T_1], H^{s+1}(\Omega))$$

with integer $s > 2$. Then there exist positive numbers $\epsilon_0 = \epsilon_0(T_)$ and $K = K(T_*)$ such that for $\epsilon \leq \epsilon_0$ the first-order system (1.3) with initial data $(\rho, \rho v, \tau_{1\epsilon}, \tau_{2\epsilon})|_{t=0}$ has a unique classical solution $(\rho^\epsilon, \rho^\epsilon v^\epsilon, \tau_{1\epsilon}^\epsilon, \tau_{2\epsilon}^\epsilon)$ satisfying*

$$(\rho^\epsilon - \rho, \rho^\epsilon v^\epsilon, \tau_{1\epsilon}^\epsilon, \tau_{2\epsilon}^\epsilon) \in C([0, T_*], H^s(\Omega))$$

and

$$\sup_{t \in [0, T_*]} \left\| [(\rho^\epsilon, \rho^\epsilon v^\epsilon, \tau_{1\epsilon}^\epsilon, \tau_{2\epsilon}^\epsilon) - (\rho, \rho v, \tau_{1\epsilon}, \tau_{2\epsilon})](\cdot, t) \right\|_s \leq K(T_*) \epsilon^2. \tag{3.2}$$

Here $\Omega = \mathbb{R}^3$ or the three-dimensional torus $[0, 1]^3$, and we are using the standard notation for Sobolev spaces, defined in [2, 5, 6, 11, 12, 13, 14].

An obvious corollary of this theorem is

Corollary 3.2. *If ρ and v possess the properties assumed in Theorem 3.1 globally in time, then the time interval where $(\rho^\epsilon, \rho^\epsilon v^\epsilon, \tau_{1\epsilon}^\epsilon, \tau_{2\epsilon}^\epsilon)$ exists goes to infinity as ϵ tends to zero.*

To see the existence claim in Theorem 3.1, we recall from the previous section that the first-order system (1.3) is symmetrizable hyperbolic. Thus, the local-in-time existence theory of regular solutions to initial-value problems of symmetrizable hyperbolic systems applies [6]. Fix $\epsilon > 0$. According to the local-in-time existence theory, there is a time interval $[0, T]$ such that (1.3) with initial data $(\rho, \rho v, \tau_{1\epsilon}, \tau_{2\epsilon})|_{t=0}$ has a unique solution $(\rho^\epsilon, \rho^\epsilon v^\epsilon, \tau_1^\epsilon, \tau_2^\epsilon)$ satisfying

$$(\rho^\epsilon - \rho, \rho^\epsilon v^\epsilon, \tau_1^\epsilon, \tau_2^\epsilon) \in C([0, T], H^s(\Omega)).$$

Note that the range G_1 of $(\rho, \rho v, \tau_{1\epsilon}, \tau_{2\epsilon})(x, t)$ satisfies $G_1 \subset\subset G$ for $\inf_{x,t} \rho(x, t) > 0$. For $G_2 \subset G$ satisfying $G_1 \subset\subset G_2$, we define

$$T^\epsilon = \sup\{T > 0 : (\rho^\epsilon - \rho, \rho^\epsilon v^\epsilon, \tau_1^\epsilon, \tau_2^\epsilon) \in H^s(\Omega), \quad (\rho^\epsilon, \rho^\epsilon v^\epsilon, \tau_1^\epsilon, \tau_2^\epsilon)(x, t) \in G_2\}.$$

Namely, $[0, T^\epsilon)$ is the maximal time interval of $H^s(\Omega)$ -existence. Note that $T^\epsilon = T^\epsilon(G_2)$ may tend to 0 as ϵ goes to 0.

In order to show that $T^\epsilon > T_*$, we exploit the convergence-stability lemma [12, 2] and only need to prove the error estimate in (3.2) for $t \in [0, \min\{T_*, T^\epsilon\})$. In this time interval, both $(\rho^\epsilon, \rho^\epsilon v^\epsilon, \tau_1^\epsilon, \tau_2^\epsilon)$ and $(\rho, \rho v, \tau_{1\epsilon}, \tau_{2\epsilon})$ are well defined, regular enough and take values in the compact set G_2 .

4. ERROR ESTIMATE

The purpose of this section is to derive the error estimate in (3.2) for $t \in [0, \min\{T_*, T^\epsilon\})$. To do this, we need some classical calculus inequalities in Sobolev spaces [4, 6].

Lemma 4.1. (i). For $s \geq 2$, $H^s = H^s(\Omega)$ is an algebra. Namely, if $f, g \in H^s$, then $fg \in H^s$ and, for all multi-indices α with $|\alpha| \leq s$,

$$\|\partial_x^\alpha(fg)\| \leq C_s \|f\|_s \|g\|_s.$$

Here C_s is a generic constant depending only on s .

(ii). For $s \geq 3$, let $f \in H^s$ and $g \in H^{s-1}$. Then for all multi-indices α with $|\alpha| \leq s$, the commutator $[\partial_x^\alpha, f]g \equiv \partial_x^\alpha(fg) - f\partial_x^\alpha g \in L^2(\Omega)$ and

$$\|[\partial_x^\alpha, f]g\| \leq C_s \|\nabla f\|_{s-1} \|g\|_{s-1}.$$

(iii). Let $f(u)$ be a smooth function of u . Then for all multi-indices α with $|\alpha| \geq 1$ we have

$$\|\partial_x^\alpha f(u)\| \leq C(|u|_\infty) \|u\|_{|\alpha|},$$

where $C(|u|_\infty)$ is a constant depending on the maximum norm $|u|_\infty$ of function $u = u(x)$.

Now we derive the error estimate. With the notation in Section 2:

$$w^\epsilon = (\rho^\epsilon, \rho^\epsilon v^\epsilon)^T, \quad z^\epsilon = (\tau_1^\epsilon, \tau_2^\epsilon)^T, \quad w_\epsilon = (\rho_\epsilon, \rho_\epsilon v_\epsilon)^T, \quad z_\epsilon = (\tau_{1\epsilon}, \tau_{2\epsilon})^T.$$

we set

$$E_1 = w^\epsilon - w_\epsilon \quad \text{and} \quad E_2 = z^\epsilon - z_\epsilon.$$

From (2.2) and (3.1) we deduce that

$$\begin{aligned} \partial_t E_1 + \sum_j (f_j(w^\epsilon) - f_j(w_\epsilon))_{x_j} + \frac{1}{\epsilon} \sum_j C_j E_{2x_j} &= 0, \\ \partial_t E_2 + \frac{1}{\epsilon} \sum_j (g_j(w^\epsilon) - g_j(w_\epsilon))_{x_j} &= -\frac{1}{\epsilon^2} S E_2 - \partial_t(\tau_{1\epsilon}, \tau_{2\epsilon})^T. \end{aligned} \tag{4.1}$$

Let α be a multi-index with $|\alpha| \leq s$. Differentiating the two sides of the last equations with ∂_x^α and setting

$$E_{1\alpha} = \partial_x^\alpha E_1 \quad \text{and} \quad E_{2\alpha} = \partial_x^\alpha E_2,$$

we obtain

$$\begin{aligned} \partial_t E_{1\alpha} + \sum_j A_j(w^\epsilon) E_{1\alpha x_j} + \frac{1}{\epsilon} \sum_j C_j E_{2\alpha x_j} &= f^\alpha, \\ \partial_t E_{2\alpha} + \frac{1}{\epsilon} \sum_j B_j(w^\epsilon) E_{1\alpha x_j} &= -\frac{1}{\epsilon^2} S E_{2\alpha} - \partial_t \partial_x^\alpha (\tau_{1\epsilon}, \tau_{2\epsilon})^T + \frac{1}{\epsilon} g^\alpha, \end{aligned} \quad (4.2)$$

where $A_j(w) = f_{jw}(w)$, $B_j(w) = g_{jw}(w)$,

$$\begin{aligned} f^\alpha &= \sum_j A_j(w^\epsilon) E_{1\alpha x_j} - \sum_j \partial_x^\alpha (f_j(w^\epsilon) - f_j(w_\epsilon))_{x_j}, \\ g^\alpha &= \sum_j B_j(w^\epsilon) E_{1\alpha x_j} - \sum_j \partial_x^\alpha (g_j(w^\epsilon) - g_j(w_\epsilon))_{x_j}. \end{aligned}$$

Set $D(w) = \eta_{ww}(w, z)$ and $H = \eta_{zz}(w, z)$. Note that $D(w)A_j(w)$ is symmetric and the identity (2.3) holds. Multiplying the first equation in (4.2) with $E_{1\alpha}^T D(w^\epsilon)$ and the second with $E_{2\alpha}^T H$, summing up the two and integrating the resultant equality over Ω gives

$$\begin{aligned} & \frac{d}{dt} \int_\Omega [E_{1\alpha}^T D(w^\epsilon) E_{1\alpha} + E_{2\alpha}^T H E_{2\alpha}] dx \\ &= -\frac{2}{\epsilon^2} \int_\Omega E_{2\alpha}^T H S E_{2\alpha} dx + 2 \int_\Omega E_{1\alpha}^T D(w^\epsilon) f^\alpha dx - 2 \int_\Omega E_{2\alpha}^T H \partial_t \partial_x^\alpha (\tau_{1\epsilon}, \tau_{2\epsilon})^T dx \\ & \quad + \frac{2}{\epsilon} \int_\Omega E_{2\alpha}^T H g^\alpha dx + \int_\Omega E_{1\alpha}^T [\partial_t D(w^\epsilon) + \sum_j \partial_{x_j} D(w^\epsilon) A_j(w^\epsilon)] E_{1\alpha} dx \\ & \quad + \frac{2}{\epsilon} \sum_j \int_\Omega E_{1\alpha}^T \partial_{x_j} D(w^\epsilon) C_j E_{2\alpha} dx \\ &\leq -\frac{c}{\epsilon^2} \|E_{2\alpha}\|^2 + \frac{3c}{4\epsilon^2} \|E_{2\alpha}\|^2 + C\epsilon^4 + C\|E_{1\alpha}\|^2 \\ & \quad + C(\|f^\alpha\|^2 + \|g^\alpha\|^2) + |\partial_t D(w^\epsilon) + \sum_j \partial_{x_j} D(w^\epsilon) A_j(w^\epsilon)|_\infty \|E_{1\alpha}\|^2 \\ & \quad + C|\sum_j \partial_{x_j} D(w^\epsilon) C_j|_\infty^2 \|E_{1\alpha}\|^2. \end{aligned} \quad (4.3)$$

Here c and C are both generic positive constants, and we have used that $\|\partial_x^\alpha \partial_t (\tau_{1\epsilon}, \tau_{2\epsilon})\| \leq C\epsilon$.

Next we analyze the terms in the last two lines. For $\|f^\alpha\|$, we use Lemma 4.1 and the boundedness of $\|w_{\epsilon x_j}\|_s$ to estimate as follows.

$$\begin{aligned}
\|f^\alpha\| &= \|\sum_j [A_j(w^\epsilon)E_{1\alpha x_j} - \partial_x^\alpha(f_j(w^\epsilon) - f_j(w_\epsilon))_{x_j}]\| \\
&\leq \sum_j \|A_j(w^\epsilon)E_{1\alpha x_j} - \partial_x^\alpha(A_j(w^\epsilon)w_{\epsilon x_j}^\epsilon - A_j(w_\epsilon)w_{\epsilon x_j})\| \\
&\leq \sum_j \|A_j(w^\epsilon)E_{1\alpha x_j} - \partial_x^\alpha(A_j(w^\epsilon)E_{1x_j} + (A_j(w^\epsilon) - A_j(w_\epsilon))w_{\epsilon x_j})\| \\
&\leq \sum_j \|[A_j(w^\epsilon), \partial_x^\alpha]E_{1x_j}\| + \sum_j \|\partial_x^\alpha((A_j(w^\epsilon) - A_j(w_\epsilon))w_{\epsilon x_j})\| \\
&\leq C_s \sum_j \|\nabla A_j(w^\epsilon)\|_{s-1} \|E_{1x_j}\|_{s-1} + C_s \sum_j \|A_j(w^\epsilon) - A_j(w_\epsilon)\|_s \|w_{\epsilon x_j}\|_s \\
&\leq C_s \sum_j \|\nabla A_j(w_\epsilon)\|_{s-1} \|E_1\|_s + C_s \sum_j \|\nabla A_j(w^\epsilon) - \nabla A_j(w_\epsilon)\|_{s-1} \|E_1\|_s \\
&\quad + C_s \sum_j \|A_j(w^\epsilon) - A_j(w_\epsilon)\|_s \\
&\leq C_s \|E_1\|_s + C_s(1 + \|E_1\|_s) \sum_j \|A_j(w^\epsilon) - A_j(w_\epsilon)\|_s \\
&\leq C_s(1 + \|E_1\|_s) \|E_1\|_s.
\end{aligned} \tag{4.4}$$

Similarly, we have

$$\|g^\alpha\| \leq C_s(1 + \|E_1\|_s) \|E_1\|_s. \tag{4.5}$$

In addition, we have

$$\begin{aligned}
|\sum_j \partial_{x_j} D(w^\epsilon) C_j| &\leq C \sum_j |w_{x_j}^\epsilon|, \\
|\partial_t D(w^\epsilon) + \sum_j \partial_{x_j} D(w^\epsilon) A_j(w^\epsilon)| &\leq C \sum_j |w_{x_j}^\epsilon| + C |w_t^\epsilon|.
\end{aligned} \tag{4.6}$$

For $|w_t^\epsilon|$ we use the equation and $z_\epsilon = O(\epsilon)$ to get

$$\begin{aligned}
|w_t^\epsilon| &\leq C \sum_j |w_{x_j}^\epsilon| + C \frac{1}{\epsilon} \sum_j |z_{x_j}^\epsilon| \\
&\leq C \sum_j (|w_{\epsilon x_j}| + |w_{x_j}^\epsilon - w_{\epsilon x_j}|) + C \frac{1}{\epsilon} \sum_j (|z_{\epsilon x_j}| + |z_{x_j}^\epsilon - z_{\epsilon x_j}|) \\
&\leq C + C \sum_j |E_{1x_j}| + C \frac{1}{\epsilon} \sum_j |E_{2x_j}| \\
&\leq C + C \|E_1\|_s + C \frac{1}{\epsilon} \|E_2\|_s.
\end{aligned} \tag{4.7}$$

Substituting (4.4)-(4.7) into (4.3) we arrive at

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega [E_{1\alpha}^T D(w^\epsilon) E_{1\alpha} + E_{2\alpha}^T H E_{2\alpha}] dx \\
&\leq -\frac{c}{4\epsilon^2} \|E_{2\alpha}\|^2 + C\epsilon^4 + C(1 + \|E_1\|_s^2 + \frac{1}{\epsilon} \|E_2\|_s) \|E_1\|_s^2.
\end{aligned}$$

Summing up this inequality over all α with $|\alpha| \leq s$ gives

$$\begin{aligned}
&\frac{d}{dt} \sum_\alpha \int_\Omega [E_{1\alpha}^T D(w^\epsilon) E_{1\alpha} + E_{2\alpha}^T H E_{2\alpha}] dx \\
&\leq -\frac{c}{4\epsilon^2} \|E_2\|_s^2 + C\epsilon^4 + C(1 + \|E_1\|_s^2 + \frac{1}{\epsilon} \|E_2\|_s) \|E_1\|_s^2 \\
&\leq -\frac{c}{4\epsilon^2} \|E_2\|_s^2 + C\epsilon^4 + C(1 + \|E_1\|_s^2) \|E_1\|_s^2 + \frac{c}{8\epsilon^2} \|E_2\|_s^2 + C \|E_1\|_s^4 \\
&\leq -\frac{c}{8\epsilon^2} \|E_2\|_s^2 + C\epsilon^4 + C(1 + \|E_1\|_s^2) \|E_1\|_s^2.
\end{aligned}$$

Integrating the last inequality from 0 to $t \in [0, \min\{T^\epsilon, T_*\})$ and using the positive definiteness of $D(w^\epsilon)$ and H , we get

$$\|E_1\|_s^2 + \|E_2\|_s^2 \leq C\epsilon^4 + C \int_0^t (1 + \|E_1\|_s^2) \|E_1\|_s^2 \equiv \phi(t). \quad (4.8)$$

Obviously, $\phi(0) = C\epsilon^4$ and

$$\phi'(t) = C(1 + \|E_1\|_s^2) \|E_1\|_s^2 \leq C\phi(1 + \phi).$$

Applying the nonlinear Gronwall-type inequality in [11] to the last inequality yields

$$\phi(t) \leq e^{CT_*}$$

for $t \in [0, \min\{T^\epsilon, T_*\})$ if we choose ϵ so small that $\phi(0) = C\epsilon^4 \leq e^{-CT_*}$. Finally, we apply the standard Gronwall inequality to (4.8) to obtain

$$\|E_1\|_s^2 + \|E_2\|_s^2 \leq \phi(t) \leq C\epsilon^4 e^{CT_*}.$$

This completes the proof.

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